Chapter 4

Mean Flow Equations

4.1 Reynolds Equations

In the previous Chapter, various statistical quantities were introduced to describe turbulent velocity fields—means, PDF's, two-point correlations, etc. It is possible to derive evolution equations for all of these quantities, starting from the Navier-Stokes equations that govern the underlying turbulent velocity field $\mathbf{U}(\mathbf{x}, t)$. The most basic of these equations (first derived by Reynolds 1894) are those that govern the mean velocity field $\langle \mathbf{U}(\mathbf{x}, t) \rangle$.

The decomposition of the velocity $\mathbf{U}(\mathbf{x}, t)$ into its mean $\langle \mathbf{U}(\mathbf{x}, t) \rangle$ and the fluctuation

$$\mathbf{u}(\mathbf{x}, t) \equiv \mathbf{U}(\mathbf{x}, t) - \langle \mathbf{U}(\mathbf{x}, t) \rangle, \tag{4.1}$$

is referred to as the *Reynolds decomposition*, i.e.,

$$\mathbf{U}(\mathbf{x}, t) = \langle \mathbf{U}(\mathbf{x}, t) \rangle + \mathbf{u}(\mathbf{x}, t). \tag{4.2}$$

It follows from the continuity equation (Eq. 2.19)

$$\nabla \cdot \mathbf{U} = \nabla \cdot (\langle \mathbf{U} \rangle + \mathbf{u}) = 0, \tag{4.3}$$

that both $\langle \mathbf{U}(\mathbf{x}, t) \rangle$ and $\mathbf{u}(\mathbf{x}, t)$ are solenoidal. For, the mean of this equation is simply

$$\nabla \cdot \langle \mathbf{U} \rangle = 0, \tag{4.4}$$

and then by subtraction we obtain

$$\nabla \cdot \mathbf{u} = 0. \tag{4.5}$$

(Note that taking the mean and differentiation commute so that $\langle \nabla \cdot \mathbf{U} \rangle = \nabla \cdot \langle \mathbf{U} \rangle$ and also $\langle \nabla \cdot \mathbf{u} \rangle = \nabla \cdot \langle \mathbf{u} \rangle = 0$.)

Taking the mean of the momentum equation (Eq. 2.35) is less simple because of the nonlinear convective term. The first step is to write the substantial derivative in conservative form,

$$\frac{\mathrm{D}U_j}{\mathrm{D}t} = \frac{\partial U_j}{\partial t} + \frac{\partial}{\partial x_i}(U_i U_j), \tag{4.6}$$

so that the mean is

$$\left\langle \frac{\mathrm{D}U_j}{\mathrm{D}t} \right\rangle = \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial}{\partial x_i}\langle U_i U_j \rangle. \tag{4.7}$$

Then, substituting the Reynolds decomposition for U_i and U_j, the nonlinear term becomes

$$\begin{aligned} \langle U_i U_j \rangle &= \langle (\langle U_i \rangle + u_i)(\langle U_j \rangle + u_j) \rangle \\ &= \langle \{ \langle U_i \rangle \langle U_j \rangle + u_i \langle U_j \rangle + u_j \langle U_i \rangle + u_i u_j \} \rangle \\ &= \langle U_i \rangle \langle U_j \rangle + \langle u_i u_j \rangle. \end{aligned} \tag{4.8}$$

For reasons soon to be given, the velocity covariances $\langle u_i u_j \rangle$ are called *Reynolds stresses.* Thus from the previous two equations we obtain

$$\begin{aligned} \left\langle \frac{\mathrm{D}U_j}{\mathrm{D}t} \right\rangle &= \frac{\partial \langle U_j \rangle}{\partial t} + \frac{\partial}{\partial x_i} \{ \langle U_i \rangle \langle U_j \rangle + \langle u_i u_j \rangle \} \\ &= \frac{\partial \langle U_j \rangle}{\partial t} + \langle U_i \rangle \frac{\partial \langle U_j \rangle}{\partial x_i} + \frac{\partial}{\partial x_i} \langle u_i u_j \rangle, \end{aligned} \tag{4.9}$$

the second step following from $\partial \langle U_i \rangle / \partial x_i = 0$ (Eq. 4.4).

The final result can be usefully re-expressed by defining the *mean substantial derivative*

$$\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t} \equiv \frac{\partial}{\partial t} + \langle \mathbf{U} \rangle \cdot \nabla. \tag{4.10}$$

For any property $Q(\mathbf{x}, t)$, $\bar{\mathrm{D}}Q/\bar{\mathrm{D}}t$ represents its rate of change following a point moving with the local mean velocity $\langle \mathbf{U}(\mathbf{x}, t) \rangle$. In term of this derivative, Eq. (4.9) is

$$\left\langle \frac{\mathrm{D}U_j}{\mathrm{D}t} \right\rangle = \frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t} \langle U_j \rangle + \frac{\partial}{\partial x_i} \langle u_i u_j \rangle. \tag{4.11}$$

Evidently the mean of the substantial derivative $\langle \mathrm{D}U_j / \mathrm{D}t \rangle$ does not equal the mean substantial derivative of the mean $\bar{\mathrm{D}} \langle U_j \rangle / \bar{\mathrm{D}}t$.

It is now a simple matter to take the mean of the momentum equation (Eq. 2.35) since the other terms are linear in **U** and p. The result is the mean momentum or *Reynolds equations*

$$\frac{\bar{\mathrm{D}}\langle U_j\rangle}{\bar{\mathrm{D}}t} = \nu\nabla^2\langle U_j\rangle - \frac{\partial\langle u_i u_j\rangle}{\partial x_i} - \frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_j}. \tag{4.12}$$

In appearance, the Reynolds equations (Eq. 4.12) and the Navier-Stokes equations (Eq. 2.35) are the same, except for the term in the Reynolds stresses—a crucial difference.

Like $p(\mathbf{x},t)$, the mean pressure field $\langle p(\mathbf{x},t)\rangle$ satisfies a Poisson equation. This may be obtained either by taking the mean of $\nabla^2 p$ (Eq. 2.42), or by taking the divergence of the Reynolds equations:

$$\begin{aligned} -\frac{1}{\rho}\nabla^2\langle p\rangle &= \left\langle \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}\right\rangle \\ &= \frac{\partial\langle U_i\rangle}{\partial x_j}\frac{\partial\langle U_j\rangle}{\partial x_i} + \frac{\partial^2\langle u_i u_j\rangle}{\partial x_i \partial x_j}. \end{aligned} \tag{4.13}$$

Exercise 4.1 Obtain from the Reynolds equations (Eq. 4.12) an equation for the rate of change of mean momentum in a fixed control volume $\mathcal{V}$ (see Fig. 4.1). Where possible express terms as integrals over the bounding control surface $\mathcal{A}$.

Exercise 4.2 For a random field $\phi(\mathbf{x},t)$, obtain the results

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} = \frac{\bar{\mathrm{D}}\phi}{\bar{\mathrm{D}}t} + \nabla\cdot(\mathbf{u}\,\phi), \tag{4.14}$$

and

$$\left\langle\frac{\mathrm{D}\phi}{\mathrm{D}t}\right\rangle = \frac{\bar{\mathrm{D}}\langle\phi\rangle}{\bar{\mathrm{D}}t} + \nabla\cdot\langle\mathbf{u}\,\phi\rangle. \tag{4.15}$$

Exercise 4.3 The mean rate-of-strain $\bar{S}_{ij}$ and mean rate-of-rotation $\bar{\Omega}_{ij}$ are defined by

$$\bar{S}_{ij} \equiv \frac{1}{2}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} + \frac{\partial\langle U_j\rangle}{\partial x_i}\right), \tag{4.16}$$

and

$$\bar{\Omega}_{ij} \equiv \frac{1}{2}\left(\frac{\partial\langle U_i\rangle}{\partial x_j} - \frac{\partial\langle U_j\rangle}{\partial x_i}\right). \tag{4.17}$$

Obtain the results:

$$\bar{S}_{ij} = \langle S_{ij} \rangle, \quad \bar{\Omega}_{ij} = \langle \Omega_{ij} \rangle, \tag{4.18}$$

$$\frac{\partial \langle U_i \rangle}{\partial x_j} \frac{\partial \langle U_j \rangle}{\partial x_i} = \bar{S}_{ij}\bar{S}_{ij} - \bar{\Omega}_{ij}\bar{\Omega}_{ij}, \tag{4.19}$$

and

$$\frac{\partial \bar{S}_{ij}}{\partial x_i} = \frac{1}{2} \frac{\partial^2 \langle U_j \rangle}{\partial x_i \partial x_i}. \tag{4.20}$$

4.2 Reynolds Stresses

Evidently the Reynolds stresses $\langle u_i u_j \rangle$ play a crucial role in the equations for the mean velocity field $\langle \mathbf{U} \rangle$. If $\langle u_i u_j \rangle$ were zero, then the equations for $\mathbf{U}(\mathbf{x}, t)$ and $\langle \mathbf{U}(\mathbf{x}, t) \rangle$ would be identical. The very different behavior of $\mathbf{U}(\mathbf{x}, t)$ and $\langle \mathbf{U}(\mathbf{x}, t) \rangle$ (see, e.g., Fig. 1.4 on page 6) is therefore attributable to the effect of the Reynolds stresses. Some of their properties are now described.

Interpretation as Stresses. The Reynolds equations can be rewritten

$$\rho \frac{\bar{\mathrm{D}} \langle U_j \rangle}{\bar{\mathrm{D}} t} = \frac{\partial}{\partial x_i} \left\{ \mu \left(\frac{\partial \langle U_i \rangle}{\partial x_j} + \frac{\partial \langle U_j \rangle}{\partial x_i} \right) - \langle p \rangle \delta_{ij} - \rho \langle u_i u_j \rangle \right\}. \tag{4.21}$$

This is the general form of a momentum conservation equation (cf. Eq. 2.31), with the term in braces representing the sum of three stresses: the viscous stress, the isotropic stress $-\langle p \rangle \delta_{ij}$ from the mean pressure field, and the apparent stress arising from the fluctuating velocity field $-\rho \langle u_i u_j \rangle$. Even though this apparent stress is $-\rho \langle u_i u_j \rangle$, it is convenient and conventional to refer to $\langle u_i u_j \rangle$ as the Reynolds stress.

The viscous stress (i.e., force per unit area) ultimately stems from momentum transfer at the molecular level. So also the Reynolds stress stems from momentum transfer by the fluctuating velocity field. Referring to Fig. 4.1, the rate of gain of momentum within a fixed control volume $\mathcal{V}$ due to flow through the bounding surface $\mathcal{A}$ is

$$\dot{\mathbf{M}} = \iint_{\mathcal{A}} \rho \mathbf{U}(-\mathbf{U} \cdot \mathbf{n}) \, \mathrm{d}A. \tag{4.22}$$

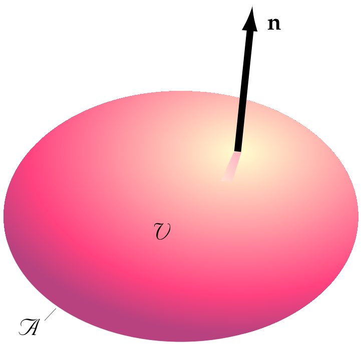

Figure 4.1: Sketch of control volume $\mathcal{V}$, with bounding control surface $\mathcal{A}$, showing the outward pointing unit normal $\mathbf{n}$.

(The momentum per unit volume is $\rho\mathbf{U}$, and the volume flow rate per unit area into $\mathcal{V}$ through $\mathcal{A}$ is $-\mathbf{U}\cdot\mathbf{n}$.) The mean of the j-component of this equation is

$$\begin{aligned}\langle \dot{M}_j \rangle &= \iint_{\mathcal{A}} -\rho\{\langle U_i\rangle\langle U_j\rangle + \langle u_i u_j\rangle\} n_i \,\mathrm{d}A \\ &= \iiint_{\mathcal{V}} -\rho \frac{\partial}{\partial x_i}\{\langle U_i\rangle\langle U_j\rangle + \langle u_i u_j\rangle\}\,\mathrm{d}V, \end{aligned} \tag{4.23}$$

the last step following from the divergence theorem. Thus, for the control volume $\mathcal{V}$, the Reynolds stress as it appears in the Reynolds equations (i.e., $-\rho\partial\langle u_i u_j\rangle/\partial x_i$) arises from the mean momentum flux due to the fluctuating velocity on the boundary $\mathcal{A}$, $-\rho\langle u_i u_j\rangle n_i$.

The Closure Problem. For a general statistically three-dimensional flow, there are four independent equations governing the mean velocity field: namely three components of the Reynolds equations (Eq. 4.12) together with either the mean continuity equation (Eq. 4.4) or the Poisson equation for $\langle p\rangle$ (Eq. 4.13). But these four equations contain more than four unknowns. In addition to $\langle\mathbf{U}\rangle$ and $\langle p\rangle$ (four quantities), there are also the Reynolds stresses.

This is a manifestation of the *closure problem.* In general, the evolution equations (obtained from the Navier-Stokes equations) for a set of statistics contains additional statistics to those in the set considered. Consequently, in the absence of separate information to determine the additional statistics, the set of equations cannot be solved. Such a set of equations—with more unknowns than equations—is said to be *unclosed.* The Reynolds equations are unclosed: they cannot be solved unless the Reynolds stresses are somehow determined.

Tensor Properties. The Reynolds stresses are the components of a second-order tensor[1], which is obviously symmetric, i.e., $\langle u_i u_j \rangle = \langle u_j u_i \rangle$. The diagonal components ($\langle u_1^2 \rangle = \langle u_1 u_1 \rangle$, $\langle u_2^2 \rangle$ and $\langle u_3^2 \rangle$) are *normal stresses*, while the off-diagonal components (e.g., $\langle u_1 u_2 \rangle$) are *shear stresses.*

The *turbulent kinetic energy* $k(\mathbf{x}, t)$ is defined to be half the trace of the Reynolds stress tensor:

$$k \equiv \tfrac{1}{2}\langle \mathbf{u} \cdot \mathbf{u} \rangle = \tfrac{1}{2}\langle u_i u_i \rangle. \tag{4.24}$$

It is the mean kinetic energy per unit mass in the fluctuating velocity field.

In the principal axes of the Reynolds stress tensor, the shear stresses are zero, and the normal stresses are the eigenvalues , which are non-negative (i.e., $\langle u_1^2 \rangle \geq 0$). Thus the Reynolds stress tensor is symmetric positive semi-definite. In general, all eigenvalues are strictly positive: but in special or extreme circumstances one or more of the eigenvalues can be zero.

Anisotropy. The distinction between shear stresses and normal stresses is dependent on the choice of coordinate system. An intrinsic distinction can be made between isotropic and anisotropic stresses. The isotropic stress is $\frac{2}{3}k\delta_{ij}$, and then the deviatoric anisotropic part is

$$a_{ij} \equiv \langle u_i u_j \rangle - \tfrac{2}{3}k\delta_{ij}. \tag{4.25}$$

The normalized anisotropytensor—used extensively below—is defined by

$$b_{ij} = \frac{a_{ij}}{2k} = \frac{\langle u_i u_j \rangle}{\langle u_\ell u_\ell \rangle} - \tfrac{1}{3}\delta_{ij}. \tag{4.26}$$

In terms of these anisotropy tensors, the Reynolds stress tensor is

$$\begin{aligned} \langle u_i u_j \rangle &= \tfrac{2}{3}k\delta_{ij} + a_{ij} \\ &= 2k(\tfrac{1}{3}\delta_{ij} + b_{ij}). \end{aligned} \tag{4.27}$$

It is only the anisotropic component a_{ij} that is effective in transporting momentum. For we have

$$\rho \frac{\partial \langle u_i u_j \rangle}{\partial x_i} + \frac{\partial \langle p \rangle}{\partial x_j} = \rho \frac{\partial a_{ij}}{\partial x_i} + \frac{\partial}{\partial x_j}(\langle p \rangle + \tfrac{2}{3}\rho k), \tag{4.28}$$

showing that the isotropic component ($\frac{2}{3}k$) can be absorbed in a modified mean pressure .

[1] The properties of second-order tensors are reviewed in Appendix B.

Irrotational Motion. An essential feature of turbulent flows is that they are rotational. Consider instead an irrotational random velocity field—such as (to an approximation) the flow of water waves. The vorticity is zero, and so in turn the mean vorticity, the fluctuating vorticity, and $\partial u_i/\partial x_j - \partial u_j/\partial x_i$ are also zero. Hence we have

$$\left\langle u_i \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \right\rangle = \frac{\partial}{\partial x_j} \tfrac{1}{2} \langle u_i u_i \rangle - \frac{\partial}{\partial x_i} \langle u_i u_j \rangle = 0, \tag{4.29}$$

from which follows the *Corrsin-Kistler equation* (Corrsin and Kistler 1954)

$$\frac{\partial}{\partial x_i} \langle u_i u_j \rangle = \frac{\partial k}{\partial x_j}, \quad \text{for irrotational flow.} \tag{4.30}$$

In this case the Reynolds stress $\langle u_i u_j \rangle$ has the same effect as the isotropic stress $k\delta_{ij}$, which can be absorbed in a modified pressure. In other words, the Reynolds stresses arising from an irrotational field $\mathbf{u}(\mathbf{x}, t)$ have absolutely no effect on the mean velocity field.

Symmetries. For some flows, symmetries in the flow geometry determine properties of the Reynolds stresses.

Consider a statistically two-dimensional flow in which statistics are independent of x_3, and which is statistically invariant under reflections of the x_3 coordinate axis. For the PDF of velocity $f(\mathbf{V}; \mathbf{x}, t)$, these two conditions imply

$$\frac{\partial f}{\partial x_3} = 0, \tag{4.31}$$

and

$$f(V_1, V_2, V_3; x_1, x_2, x_3, t) = f(V_1, V_2, -V_3; x_1, x_2, -x_3, t). \tag{4.32}$$

At $x_3 = 0$, this last equation yields $\langle U_3 \rangle = -\langle U_3 \rangle$, i.e., $\langle U_3 \rangle = 0$: it similarly yields $\langle u_1 u_3 \rangle = 0$ and $\langle u_2 u_3 \rangle = 0$. And the first equation (Eq. 4.31) indicates that these relations hold for all $\mathbf{x}$. Thus, for such a statistically two-dimensional flow , $\langle U_3 \rangle$ is zero and the Reynolds-stress tensor is

$$\begin{bmatrix} \langle u_1^2 \rangle & \langle u_1 u_2 \rangle & 0 \\ \langle u_1 u_2 \rangle & \langle u_2^2 \rangle & 0 \\ 0 & 0 & \langle u_3^2 \rangle \end{bmatrix}. \tag{4.33}$$